Degree-constrained orientations of embedded graphs

Yann Disser* Jannik Matuschke†

Preprint 032-2011
December 22, 2011
(Revised: February 23, 2012)

We consider the problem of orienting the edges of an embedded graph in such a way that the in-degrees of both the nodes and faces meet given values. We show that the number of feasible solutions is bounded by 2^{2g}, where g is the genus of the embedding, and all solutions can be determined within time $\mathcal{O}(2^{2g}|E|^2 + |E|^3)$. In particular, for planar graphs the solution is unique if it exists and for every fixed genus there is a polynomial time algorithm to find all solutions. We show that the problem becomes NP-complete if only upper and lower bounds on the in-degrees are specified instead of exact values.

1 Introduction

Graph orientation is an area of combinatorial optimization that deals with the problem of assigning directions to the edges of an undirected graph, subject to certain problem-specific requirements. Besides yielding useful structural insights, e.g., with respect to connectivity of graphs [14] and hypergraphs [10], research in graph orientation is motivated by applications in areas such as graph drawing [2, 4] or efficient data structures for planar graphs [3].

A particularly well-studied class of orientation problems are degree-constrained problems, i.e., where the in-degree of each vertex in the resulting orientation has to lie within certain bounds. Hakimi [11] and Frank [9] provided good characterizations[1] for the existence of such orientations. In this paper, we answer a question raised by András

*ETH Zurich, Intitute of Theoretical Computer Science, e-mail: ydisser@inf.ethz.ch
†TU Berlin, Institut für Mathematik, e-mail: matuschke@math.tu-berlin.de
Partly supported by Berlin Mathematical School.
[1] A good characterization of a decision problem in the sense of Edmonds [5] is a description of polynomially verifiable certificates for both yes- and no-instances of the problem.

Frank [8], asking for a good characterization for the following problem: Given an embedding of a graph in the plane, is there an orientation of the edges that meets prescribed in-degrees both in the primal and the dual graph at the same time? We show that if such an orientation exists, it is unique and can be computed by combining a feasible orientation for the primal graph with a feasible orientation for the dual graph. Our result generalizes to graph embeddings of higher genus, showing that the number of feasible orientations is bounded by a function of the genus, and the set of all solutions can be computed efficiently as long as the genus is fixed. We also show that the problem becomes NP-complete as soon as upper and lower bounds on the in-degrees are specified instead of exact values.

Related work Research in graph orientation has a long history that revealed many interesting structural insights and applications. E.g., a classical result by Robbins [14] states that an undirected graph is 2-edge-connected if and only if it has an orientation that is strongly connected. This result was translated to hypergraphs by Frank et al. [10]. Graph orientation is also closely connected to graph drawing. For example, Eades and Wormald [4] showed hardness of a fixed edge-length graph drawing problem using an orientation problem on a planar graph as an important device in their reduction. More recently, Biedl et al. [2] provided a 13/8-approximation algorithm for finding a balanced acyclic orientation, with implications for orthogonal graph drawing.

Regarding degree-constrained orientation problems, Hakimi [11] gave a good characterization for the existence of orientations that match given prescribed in-degrees exactly and also for the orientations that fulfill either lower or upper bounds on the in-degrees. Frank [9] observed that the results for lower and upper bounds can easily be combined in a constructive way to find orientations that fulfill upper and lower bounds at the same time. Asahiro et al. [1] consider an optimization version of the degree-constrained orientation problem where a penalty function on the violated degree-bounds is to be minimized. They find that the problem is solvable in polynomial time if the penalty function is convex, but APX-hard in case of concave penalty functions.

Orientations of planar graphs received special attention by the research community because they revealed several interesting properties. Based on the insight that every planar graph allows for an orientation with maximum in-degree 3, Chrobak and Eppstein [3] designed a highly efficient data structure for adjacency queries in planar graphs. In a distinct line of research, Felsner [6] showed that the set of orientations fulfilling a prescribed in-degree in a planar graph carries the structure of a distributive lattice.

Contribution and structure of the paper In this paper, we consider an extension of the degree-constrained problem, which we call *primal-dual orientation problem.* The input to this problem is an embedding of a graph in a surface and we require in-degree prescriptions not only to be met for every vertex but also for every face of the embedding. This variant of the problem was first proposed by András Frank for the special case of plane graphs [8] in conjunction with the question for a good characterization of the existence of such an orientation.

Before we present our results, we give a short introduction to orientations and embedded graphs in Section 2. Section 3 then deals with the primal-dual orientation problem with fixed in-degrees and contains two different proofs that yield the answer to Frank's question. Subsection 3.1 comprises a combinatorial proof for the uniqueness of the solution in plane graphs, also reducing the problem to solving the original degree-constrained orientation problem once in the primal and once in the dual graph. In Subsection 3.2, an alternative proof based on a simple linear algebra argument also yields a bound on the number of feasible orientations in embeddings of higher genus. In Section 4, we show that if we accept bounds on the in-degrees instead of exact values, the problem becomes NP-complete. In Section 5, we point out an open question, which will be subject of future research.

2 Preliminaries

We give a short introduction on graph embeddings and orientations of those embeddings. Throughout this paper we will assume all graphs to be connected but not necessarily simple, i.e., loops and multi-edges are allowed. While the connectedness assumption is very common in the context of graph embeddings, all results presented here can be extended to non-connected graphs by temporarily introducing additional edges (and adjusting the in-degree specifications acordingly) so as to render the graph connected.

Embedded graphs An *embedding* of a graph is a mapping of the vertices and edges of the graph onto a closed surface (e.g., a sphere or a torus) such that edges meet only at common vertices. This mapping partitions the surface into several regions, called *faces*. The *dual* of an embedded graph is the graph that is obtained by the following procedure: For every face in the embedding, introduce a vertex in the dual graph. For every edge of primal graph, introduce an edge in the dual graph that connects the faces that are adjacent to the original edge. The genus g of the embedding is determined by Euler's formula: If E is the set of edges, V is the set of vertices and V^* is the set of faces, then $|V| + |V^*| - |E| = 2 - 2g$.

If $g = 0$, i.e., the graph is embedded in a sphere, the embedding is called *planar* (as embeddings on spheres and planes are combinatorially equivalent). Planar embeddings have several features that make them particularly interesting. In this work, we will make use of the following fact, called *cycle-cut duality* [15], which holds (exclusively) in planar embeddings: A set of edges is a simple cycle in the primal if and only if it is a simple cut in the dual and vice versa.

More details on embedded graphs can be found in [12].

Orientations of primal and dual graphs An orientation of a graph is an assignment of directions to the edges, i.e., for every edge we specify one of the two endpoints of the edge as its head and the other as its tail. By convention, we orient the edges in the dual graph in such a way that they cross their primal "alter egos" from right to left (cf. Figure 1). Thus every orientation of the primal graph induces an orientation of the dual

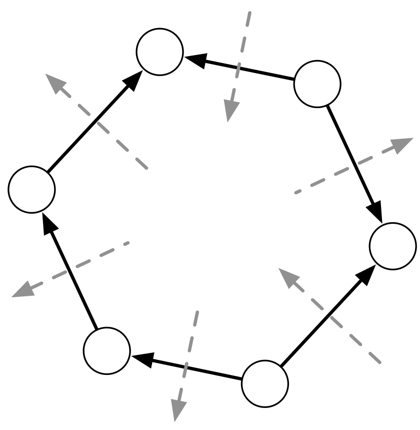

Figure 1: Induced orientations of the edges in the dual graph. An edge in the dual graph crosses its corresponding edge in the primal graph from right to left.

graph and vice versa. Given an orientation D, we denote the set of edges whose head is the vertex v by $\delta_D^-(v)$ and the set of edges whose tail is v by $\delta_D^+(v)$. In accordance with our convention for dual orientations, we let $\delta_D^-(f)$ be the set of edges whose left face is f, and $\delta_D^+(f)$ be the set of edges whose right face is f.

We mention that our convention for primal and dual orientations extends cycle/cut duality in the sense that a directed simple cycle in the primal is a directed simple cut in the dual and vice versa.

3 Orientations with fixed in-degrees

We consider the problem of finding an orientation that meets given fixed in-degrees for both the vertices and faces of the embedded graph, called the primal-dual orientation problem. We start this section by stating a formal description of the problem.

Problem. *(Primal-dual orientation problem)*

Given: *an embedded graph* $G = (V, E)$*, two functions* $\alpha : V \to \mathbb{N}_0$*,* $\alpha^* : V^* \to \mathbb{N}_0$

Task: *Find an orientation* D *of the edges* E *such that* $|\delta_D^-(v)| = \alpha(v)$ *for all* $v \in V$ *and* $|\delta_D^-(f)| = \alpha^*(f)$ *for all* $f \in V^*$*, or prove that there is none.*

Primal and dual feasibility The following notation will be useful throughout the proofs in this section. Given an instance of the primal-dual orientation problem, we call an orientation D

- *primally feasible* if $|\delta_D^-(v)| = \alpha(v)$ for all $v \in V$.

- *dually feasible* if $|\delta_D^-(f)| = \alpha^*(f)$ for all $f \in V^*$.
- *globally feasible* if it is primally and dually feasible.

The primal-dual orientation problem thus asks for a globally feasible orientation. It is clear that the existence of both primally feasible solutions and dually feasible solutions is necessary for the existence of such a globally feasible orientation. However, it can easily be checked that this is not sufficient: For example, consider a planar graph with two vertices and two parallel edges connecting them, and let $\alpha(v) = 1$ and $\alpha^*(f) = 1$ for all $v \in V$ and $f \in V^*$. While orienting both edges in opposite directions in the primal graph is primally feasible, orienting them in the same direction (which is orienting them in oposite directions in the dual graph) is dually feasible. However, none of the orientations is globally feasible.

In this section, we will present two approaches for obtaining necessary and sufficient conditions for existence of globally feasible solutions.

3.1 A combinatorial approach for planar embeddings

In this section we want to provide a combinatorial argument for the uniqueness of a feasible solution to the primal-dual orientation problem in the planar case. We show how to construct a globally feasible solution from an orientation that is feasible in the primal graph and an orientation that is feasible in the dual graph.

Rigid edges Hakimi [11] showed that a primally feasible orientation exists if and only if $\sum_{v \in V} \alpha(v) = |E|$ and $\sum_{v \in S} \alpha(v) \geq |E[S]|$ for all $S \subseteq V$, where $E[S]$ is the set of edges with both endpoints in S. The necessity follows from the fact that every edge in $E[S]$ contributes to the in-degree of a node in S, independent of its orientation.

Now consider a subset $S \subseteq V$ with $\sum_{v \in S} \alpha(v) = |E[S]|$. All edges that have one end point in S and one end point in $V \setminus S$ must be oriented from S to $V \setminus S$ in all primally feasible orientations. We call edges whose orientation is fixed in this way *primally rigid*[2] and denote the set of all primally rigid edges by R. Analogously, we define the set of *dually rigid* edges R^* as those that are fixed for all dually feasible orientations due to a tight set $S^* \subseteq V^*$ of faces with $\sum_{f \in S^*} \alpha^*(f) = |E[S^*]|$. It is easy to check that an edge is primally rigid if and only if it is on a directed cut in the primal graph with respect to any primally feasible orientation. Likewise, an edge is dually rigid if it is on a directed cut in the dual graph with respect to any dually feasible orientation. Furthermore, note that the set of edges on directed cuts is invariant for all feasible solutions.

Our main result in this section follows from this characterization of rigid edges and the duality of cycles and cuts in planar graphs.

Theorem 1. *In case of a planar embedding, there exists a globally feasible orientation if and only if the following three conditions are fulfilled.*

(1) There exists both a primally feasible orientation D and a dually feasible orientation D^*.

[2]The term "rigid" for edges on a directed cut of an orientation is taken from [6].

(2) The edge set can be partitioned into primally and dually rigid edges ($E = R \dot{\cup} R^*$).

(3) The orientation obtained by orienting all primally rigid edges in the same direction as they are oriented in D and all dually rigid edges in the same orientation as they are oriented in D^* is globally feasible.

If it exists, the solution is unique.

Proof. The sufficiency of the conditions is trivial, as the third condition requires the existence of a globally feasible orientation.

In order to show necessity, assume there exists a globally feasible orientation D_0. As D_0 is both primally and dually feasible, it fulfills Condition (1) of the theorem. An edge is primally rigid if and only if it is on a directed cut (w.r.t. D_0) in the primal graph. It is dually rigid, if and only if it is on a directed cut in the dual graph. Thus, by cycle/cut duality of planar graphs, an edge is dually rigid if and only if it is on a directed cycle in the primal graph. As every edge in the primal graph is either on a directed cut or on a directed cycle, the sets of primally and dually rigid edges comprise a partition of E, proving Condition (2). Now, let D be a primally feasible orientation and D^* be a dually feasible orientation. As D_0 equals D on all primally rigid edges and equals D^* on all dually rigid edges, the construction described in Condition (3) yields D_0 and is feasible.

As all edges are either primally of dually rigid, they must have the same orientation in all globally feasible solutions, and D_0 is unique. □

Note that the globally feasible solution constructed in the third condition does not depend on the choice of D and D^*. As primally and dually feasible solutions can be found in polynomial time, and rigid edges can be identified by determining the strongly connected components with respect to D and D^*, respectively, Theorem 1 yields a polynomial time algorithm to solve the problem for planar embeddings.

Corollary 2. *The primal-dual orientation problem in planar embeddings can be solved in time $\mathcal{O}(|E|^2)$.*

Proof. By Theorem 1, the problem can be solved by computing a primally feasible solution and a dually feasible solution and identifying the corresponding rigid edges. A primally feasible orientation can be found in time $\mathcal{O}(|V||E|)$ by using a simple push/relabel type algorithm [7]. Now applying the same result to the dual gives a total time of $\mathcal{O}((|V| + |V^*|) \cdot |E|) = \mathcal{O}(|E|^2)$ for determining the two orientations. Identifying directed cuts is equivalent to identifying strongly connected components, which can be done in time $\mathcal{O}(|E|)$. □

3.2 A linear algebra analysis for general embeddings

The primal-dual orientation problem can be formulated as a system of linear equalities over binary variables. To this end, we fix an arbitrary orientation D of the graph and introduce for every edge $e \in E$ a decision variable $x(e)$ that determines whether the orientation of the edge should be reversed (if it is 1) or not (if it is 0) in order to become

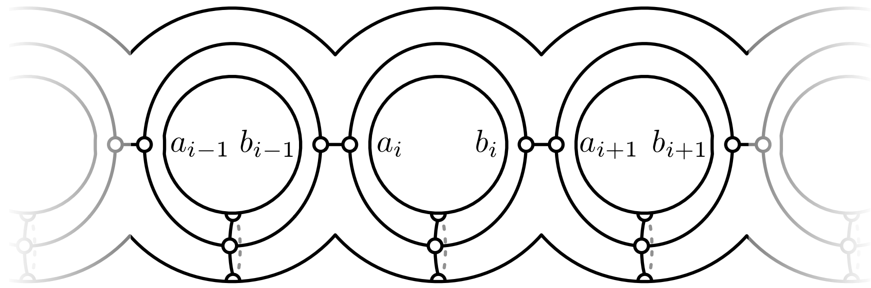

Figure 2: Construction of an instance with 2^{2g} feasible orientations, showing the tightness of the bound in Theorem 3. The base graph consists of two cycles of length 3 intersecting in a common vertex and is embedded in a torus. Examples of genus g are obtained by introducing g copies of the base graph.

globally feasible. The vector $x \in \{0,1\}^E$ yields a feasible orientation if and only if it satisfies the following system of equalities:

$$\begin{aligned} \sum_{e \in \delta_D^+(v)} x(e) - \sum_{e \in \delta_D^-(v)} x(e) &= \alpha(v) - |\delta_D^-(v)| \quad && \forall v \in V \\ \sum_{e \in \delta_D^+(f)} x(e) - \sum_{e \in \delta_D^-(f)} x(e) &= \alpha^*(f) - |\delta_D^-(f)| \quad && \forall f \in V^* \end{aligned}$$

The matrix corresponding to the first set of equalities is the incidence matrix of the primal graph, while the matrix corresponding to the second type of equalities is the incidence matrix of the dual graph (both graphs directed according to the orientation D). As we assume the graph to be connected, we know that the rank of the former matrix is $|V|-1$, while the rank of the latter matrix is $|V^*|-1$. Using the fact that the boundary of a face is a closed walk in the primal graph, it is easy to see that the rows of the first matrix are orthogonal to the rows of the second matrix. This implies that all feasible solutions are contained in a subspace of $\mathbb{R}^E$ of dimension $|E|-|V|-|V^*|+2 = 2g$.

Theorem 3. *There are at most 2^{2g} distinct solutions to the primal-dual orientation problem. The set of all globally feasible orientations can be determined in time $\mathcal{O}(2^{2g}|E|^2 + |E|^3)$. The bound on the number of orientations is tight, i.e., there are embedded graphs of genus g that allow for 2^{2g} distinct orientations.*

Proof. By basis augmentation, there is a set $A \subseteq E$ of $2g$ edges such that adding equalities $x(e) = a(e)$ with $a(e) \in \{0,1\}$ for all $e \in A$ results in a system with full rank, i.e., it has at most one solution. If for some $a \in \{0,1\}^A$ the unique solution exists and is a 0-1-vector, it corresponds to the unique globally feasible orientation that orients the edges of A according to the values $a(e)$. Otherwise, there is no such globally feasible orientation. Thus, solving the equality system for all $|\{0,1\}^A| = 2^{2g}$ possible values of a yields all possible solutions to the primal-dual orientation problem. This takes time $\mathcal{O}(|E|^3)$ for inverting the $|E|\times|E|$-matrix and $\mathcal{O}(2^{2g}|E|^2)$ for multiplying the 2^{2g} distinct right hand side vectors.

To see that the bound on the number of orientations is tight, consider the example depicted in Figure 2. The example is constructed from a base graph consisting of two cycles of length 3 sharing a common vertex. The base graph is embedded in a torus, thus featuring only a single face f. When setting $\alpha^* = |E| = 6$, any orientation is dually feasible as all dual edges are self-loops. We set the in-degree specification to 2 for the vertex at the intersection of the cycles and to 1 for the other vertices. Now, an orientation of the base graph is primally feasible, if and only if the edges of each cycle are orientented in the same direction. As the two cycles can be oriented independently, the base graph has 4 feasible orientations.

Examples of higher genus can be obtained by introducing g copies of the embedding described above. The graphs are joined via an edge from node b_i to a_{i+1} for $i \in \{1, \ldots, g-1\}$. The resulting embedding has $5g$ vertices and $7g - 1$ edges and still has only a single face. We increase the in-degree specifications of each base graph by setting $\alpha(a_{i+1}) = 2$ for $i \in \{1, \ldots, g-1\}$, so that the new edges joining the copies have to be oriented from copy i to copy $i+1$. The in-degree specification of the face is set to $|E| = 7g - 1$. Now each copy of the base graph still has its 4 feasible orientations, so in total there are 4^g feasible orientations.[3] □

4 Orientations with upper and lower bounds

A generalization of the primal-dual orientation problem asks for an orientation that fulfills upper and lower bounds on the in-degrees of vertices and faces instead of attaining fixed values. We show that this problem becomes NP-complete, even when restricted to instances with embeddings of a fixed genus (e.g., planar graphs).

Problem. *(Bounded primal-dual orientation problem)*

Given: *an embedded graph $G = (V, E)$, functions $\alpha, \beta : V \to \mathbb{N}_0$, $\alpha^*, \beta^* : V^* \to \mathbb{N}_0$*

Task: *Find an orientation D of the edges E such that $\alpha(v) \leq |\delta_D^-(v)| \leq \beta(v)$ for all $v \in V$ and $\alpha^*(f) \leq |\delta_D^-(f)| \leq \beta^*(f)$ for all $f \in V^*$, or prove that there is none.*

Theorem 4. *The bounded primal-dual orientation problem is NP-complete for graphs with any fixed genus.*

Proof. An orientation that solves the bounded primal-dual orientation problem can easily be verified in polynomial time. Hence, it remains to show that the problem is NP-hard. It is sufficient to do this for planar graphs. We use a reduction from planar 3-SAT, which is known to be an NP-hard problem [13]. In the following, we let $G_{3\text{SAT}}$ denote a fixed

[3] Note that while the primal graph in the construction described above could also be embedded in a plane, this can be avoided by introducing additional vertices and edges.

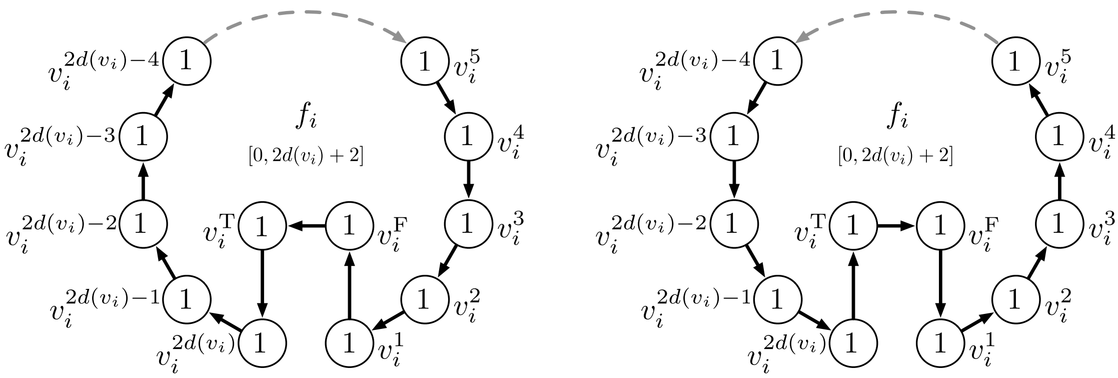

Figure 3: Illustration of the variable gadget for a variable v_i. The gadget admits only the depicted orientations, the one on the left is interpreted as v_i being 'true' and the other as v_i being 'false'.

embedding of the planar graph corresponding to a given instance of planar 3-SAT. We proceed to construct an instance $(G, \alpha, \beta, \alpha^*, \beta^*)$ of the bounded primal-dual orientation problem that has a solution if and only if the instance of planar 3-SAT has a solution. The construction consists of three parts: a *variable gadget* for each variable in $G_{3\mathrm{SAT}}$, a *clause gadget* for each clause in $G_{3\mathrm{SAT}}$, and an *edge gadget* for each edge in $G_{3\mathrm{SAT}}$.

For each variable v_i of degree $d(v_i)$ in $G_{3\mathrm{SAT}}$ we introduce a variable gadget (cf. Figure 3). The gadget consists of a cycle of length $2d(v_i)+2$, and we refer to the vertices in this cycle as $v_i^{\mathrm{T}}, v_i^{\mathrm{F}}, v_i^1, v_i^2, \dots, v_i^{2d(v_i)}$. The construction induces a single face which we call f_i. We set $\alpha^*(f_i) = 0, \beta^*(f_i) = 2d(v_i) + 2$, $\alpha(v_i^{\mathrm{T}}) = \beta(v_i^{\mathrm{T}}) = \alpha(v_i^{\mathrm{F}}) = \beta(v_i^{\mathrm{F}}) = 1$. For now, in order to understand the idea behind the variable gadget, we set $\alpha(v) = \beta(v) = 1$ for every $v \in \{v_i^1, v_i^2, \dots, v_i^{2d(v_i)}\}$, but we will change this when extending the construction later. Let us analyze the construction so far. Since every vertex requires an in-degree of exactly 1, all edges of the cycle need be oriented the same with respect to f_i, i.e., only two orientations of the gadget are permitted. We interpret each of the two possible orientations as a truth assignment for the variable v_i, depending on the direction of the edge between v_i^{T} and v_i^{F}. Directing the edge towards v_i^{T} is interpreted as setting v_i to 'true', and directing it towards v_i^{F} is interpreted as setting v_i to 'false'.

For each clause C_j in $G_{3\mathrm{SAT}}$ we introduce a clause gadget that is a cycle with 9 vertices $c_j^1, c_j^{1,\mathrm{F}}, c_j^{1,\mathrm{T}}, c_j^2, c_j^{2,\mathrm{F}}, c_j^{2,\mathrm{T}}, c_j^2, c_j^{2,\mathrm{F}}, c_j^{2,\mathrm{T}}$ enclosing a face F_j (cf. Figure 4). We set $\alpha(c_j^\ell) = \beta(c_j^\ell) = 2$ for $\ell \in \{1, \dots, 3\}$ and $\alpha^*(F_j) = 4, \beta^*(F_j) = 6$. We will set the bounds for the remaining vertices later. For now, observe that any valid orientation has to direct the edges incident to c_j^1, c_j^2, c_j^3 towards these vertices. Each of the three remaining edges can be oriented either way, provided that at least one is in counter-clockwise orientation relative to the face F_j. For each $\ell \in \{1, \dots, 3\}$, the edge $(c_j^{\ell,\mathrm{F}}, c_j^{\ell,\mathrm{T}})$ will determine the truth assignment to one literal of C_j: if it is directed towards $c_j^{\ell,\mathrm{F}}$, the corresponding literal is considered 'false', otherwise it is considered 'true'. In these terms, our construction enforces that at least one literal of C_j has to be 'true'.

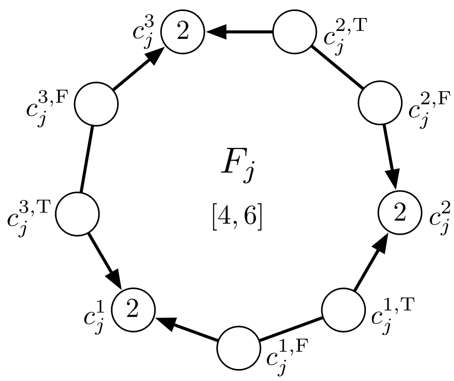

Figure 4: Illustration of the clause gadget for a clause C_j. All directed edges are rigid, and the orientation of each of the three remaining edges represents a truth assignment to a literal of the clause. At least one of these three edges needs to be oriented counter-clockwise with respect to F_j.

So far, we have provided a construction for each variable that can be oriented in two ways only, and we have given an interpretation of this orientation as a truth assignment to the variable. Also, we have provided a construction for each clause together with an interpretation of each valid orientation as a truth assignment to the literals of the clause. What remains is to show how to connect the two constructions in a way that guarantees consistency of the truth assignments to variables and literals. To this purpose, we introduce an edge gadget for each edge e_{ij} in $G_{3\mathrm{SAT}}$ between variable v_i and clause C_j as follows (cf. Figure 5). We assume a fixed counter-clockwise ordering of the edges at each vertex in the embedding of $G_{3\mathrm{SAT}}$. Suppose that e_{ij} is the k-th edge at v_i and the l-th edge at C_j with respect to this ordering. We introduce an additional edge between v_i^{2k-1} and v_i^{2k} and set $\alpha^*(f) = \beta^*(f) = 1$ for the new face f. We reassign $\alpha(v_i^{2k}) = \beta(v_i^{2k}) = \alpha(v_i^{2k-1}) = \beta(v_i^{2k-1}) = 2$ and set $\alpha(c_j^{l,\mathrm{F}}) = \beta(c_j^{l,\mathrm{F}}) = \alpha(c_j^{l,\mathrm{T}}) = \beta(c_j^{l,\mathrm{T}}) = 1$. The remaining construction depends on whether v_i appears in a positive or negative literal in C_j. If v_i appears in a positive literal, we add two vertices w_{ij}^1, w_{ij}^2 connected by two edges with $\alpha^*(f) = \beta^*(f) = 1$ for the induced face f. We add the edges $(v_i^{2k}, w_{ij}^1), (w_{ij}^1, c_j^{l,\mathrm{F}}), (v_i^{2k-1}, w_{ij}^2), (w_{ij}^2, c_j^{l,\mathrm{T}})$, which yields two additional faces f_1, f_2. We set $\alpha(w_{ij}^1) = \beta(w_{ij}^1) = \alpha(w_{ij}^2) = \beta(w_{ij}^2) = 2$, $\alpha^*(f_1) = \alpha^*(f_2) = 0$, and $\beta^*(f_1) = \beta^*(f_2) = 4$. Observe that in any valid orientation, the edge $(c_j^{l,\mathrm{F}}, c_j^{l,\mathrm{T}})$ is directed towards $c_j^{l,\mathrm{T}}$ (i.e., the corresponding literal is 'true') if and only if v_i is 'true'. Now, if v_i appears in a negative literal, we instead simply add the two edges $(v_i^{2k}, c_j^{l,\mathrm{F}}), (v_i^{2k-1}, c_j^{l,\mathrm{T}})$. This yields an additional face f, for which we set $\alpha^*(f) = 0, \beta^*(f) = 4$. Observe that in any valid orientation, the edge $(c_j^{l,\mathrm{F}}, c_j^{l,\mathrm{T}})$ is directed towards $c_j^{l,\mathrm{T}}$ (i.e., the corresponding literal is 'true') if and only if v_i is 'false'. Figure 6 shows an example of the complete construction for a 3-SAT instance.

The above construction admits an orientation if and only if the corresponding instance

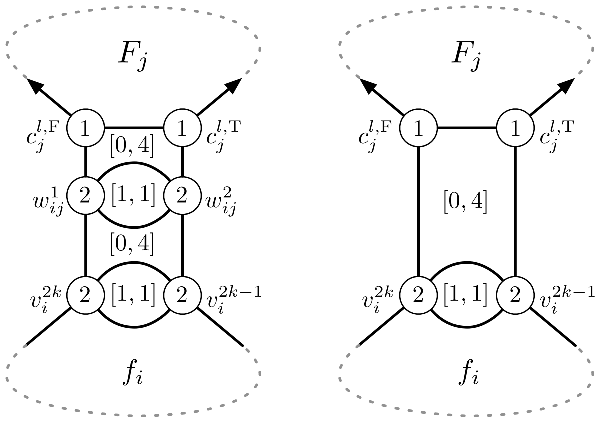

Figure 5: Illustration of the edge gadget for an edge connecting variable v_i with clause C_j. The gadget on the left is used when v_i appears in a positive literal in C_j, and the one on the right is used when v_i appears in a negative literal.

of planar 3-SAT admits a satisfying truth assignment. If it exists, the truth assignment can easily be inferred from the orientation by the interpretation given above. Finally, the construction can be made in polynomial time, which concludes our reduction. □

Corollary 5. *The bounded primal-dual orientation problem is NP-complete even when restricted to instances with $\alpha = \beta$ or $\alpha^* = \beta^*$.*

Proof. This follows from the fact that the construction in the proof of Theorem 4 has $\alpha = \beta$. By duality, the reduction can also be achieved by an instance with $\alpha^* = \beta^*$. □

5 Conclusion and open questions

We have shown that the primal-dual orientation problem in an embedded graph of genus g has at most 2^g feasible solutions and the set of all solutions can be computed in time $\mathcal{O}(2^{2g}|E|^2 + |E|^3)$. In particular, the solution is unique if the embedding is planar. However, the problem becomes NP-complete immediately, if only upper and lower bounds on the in-degrees are specified.

While these results give a relatively clear characterization of the complexity of the primal-dual orientation problem, we still want to point out an open question resulting from our research: The algorithm proposed in the proof of Theorem 3 has a running time that is exponential in the genus of the embedding. Is it possible to devise an algorithm that finds a globally feasible orientation in time polynomial in the genus of the embedding?

Acknowledgement We thank Kristóf Bérczi and Júlia Pap for providing many helpful suggestions.

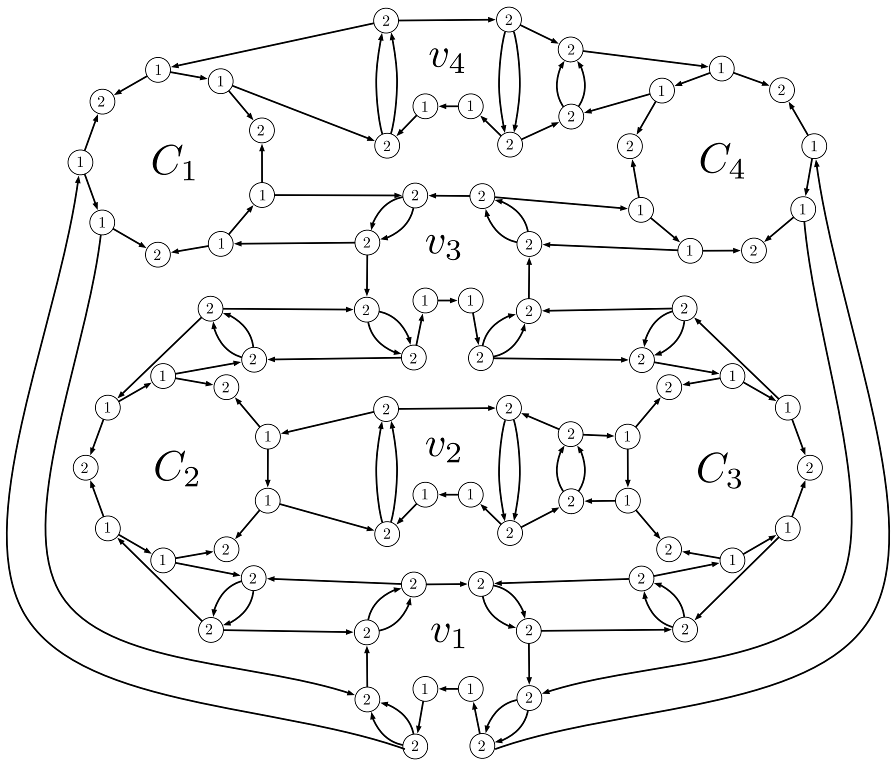

Figure 6: Example for the reduction of a 3-SAT instance $(\neg v_1 \vee \neg v_3 \vee \neg v_4) \wedge (v_1 \vee \neg v_2 \vee v_3) \wedge (v_1 \vee v_2 \vee v_3) \wedge (\neg v_1 \vee \neg v_3 \vee v_4)$ with four clauses and four variables. The orientation corresponds to the assignment $v_1 \to 1$, $v_2 \to 1$, $v_3 \to 0$, $v_4 \to 1$.

References

[1] Y. Asahiro, J. Jansson, E. Miyano, and H. Ono. Upper and lower degree bounded graph orientation with minimum penalty. In *Proceedings of the 18th Computing: The Australasian Theroy Symposium*, to appear.

[2] T. Biedl, T. Chan, Y. Ganjali, M.T. Hajiaghayi, and D.R. Wood. Balanced vertex-orderings of graphs. *Discrete Applied Mathematics*, 148(1):27–48, 2005.

[3] M. Chrobak and D. Eppstein. Planar orientations with low out-degree and compaction of adjacency matrices. *Theoretical Computer Science*, 86(2):243–266, 1991.

[4] P. Eades and N.C. Wormald. Fixed edge-length graph drawing is NP-hard. *Discrete Applied Mathematics*, 28(2):111–134, 1990.

[5] J. Edmonds. Minimum partition of a matroid into independent subsets. *Journal of Research National Bureau of Standards Section B*, 69:67–72, 1965.

[6] S. Felsner. Lattice structures from planar graphs. *Journal of Combinatorics*, 11(1):15, 2004.

[7] A. Frank. *Connections in combinatorial optimization.* Oxford University Press, 2011.

[8] A. Frank. personal communication, February 2010.

[9] A. Frank and A. Gyárfás. How to orient the edges of a graph. *Colloquia mathematica societatis Janos Bolyai*, 18:353–364, 1976.

[10] A. Frank, T. Király, and Z. Király. On the orientation of graphs and hypergraphs. *Discrete Applied Mathematics*, 131(2):385–400, 2003.

[11] S.L. Hakimi. On the degrees of the vertices of a directed graph. *Journal of the Franklin Institute*, 279(4):290–308, 1965.

[12] P. Klein. Planar graph algorithms. Lecture at Brown University, Providence, 2009, available from http://www.cs.brown.edu/courses/cs250/.

[13] D. Lichtenstein. Planar formulae and their uses. *SIAM Journal on Computing*, 11(2):329–343, 1982.

[14] H. Robbins. A theorem on graphs, with an application to a problem of traffic control. *The American Mathematical Monthly*, 46(5):281–283, 1939.

[15] H. Whitney. Non-separable and planar graphs. *Transactions of the American Mathematical Society*, 34(2):339–362, 1932.